Conformal scattering and nonlinear equations.

Subject proposed by Jean-Philippe Nicolas (Brest) and Lionel Mason (Oxford)

In the 1960’s, Roger Penrose introduced the notion of conformal compactification. The idea was to propose a geometrical method for the asymptotic analysis of field equations on a space-time (covariant hyperbolic equations). He used this approach to give a very simple formulation of the peeling property, a type of asymptotic behaviour of zero rest-mass fields discovered by Sachs and initially formulated in a complicated manner as the alignment of the principal null directions of the terms of an asymptotic expansion of the field along a null geodesic. Penrose proposed very early to use the conformal method for the construction of a scattering theory for the Einstein equations: one fixes data at past null infinity in order to solve the characteristic Cauchy problem for the Einstein equations and one propagates the solution up to future null infinity. This construction in itself is not yet a scattering theory, it remains to establish that one can then perform the construction the other way around, in other words, that the map between the scattering data at null infinity and the resulting spacetime is one-to-one.

Sofar, Penrose’s ideas have not been applied to the Einstein equations, except for a recent construction due to Dafermos and Rodnianski in which they only perform the first part of the construction (which is already remarkable) but do not go as far as establishing a complete scattering theory. However in 1980, Friedlander [3] applied them to the wave equation on static Lorentzian manifolds admitting a smooth conformal compactification. Baez and his collaborators (see e.g. [1]) extended Friedlander’s work to nonlinear equations, still on static backgrounds. In the early 2000’s, Lionel Mason and myself [7] extended the conformal scattering construction to generically non-stationary situations, adapting Hörmander’s method [4] for solving the characteristic Cauchy problem. A few years later, I adapted our work to the Schwarzschild metric, which is static, but contains a black hole and therefore has a radically different geometry at timelike infinity. My student Jérémie Joudioux [5, 6] established analogous results to [7] for a semi-linear wave equation. Very recently, conformal methods have been used by Grigalius Tajuanskas to study the scattering of the Maxwell-wave system, that is a nonlinear system with gauge freedom, on spacetimes with a different asymptotic structure ([9] as well as a work in progress in collaboration with Lionel Mason and myself).

Although analytic scattering theory on black hole spacetimes has been extensively studied, there are very few results on conformal scattering for such backgrounds. The difficulty is the singularity at timelike infinity and the large amount of work necessary to control the conformal field in its neighbourhood. On the other hand, the recent studies of nonlinear equations, even if they are an important step forward for the theory, do not tackle the delicate problem of charges at infinity, i.e. the case where the solutions generically do not decay fast enough at infinity to remain smooth at the conformal boundary.

This thesis will be co-supervised between Brest and Oxford, the two supervisors being Lionel Mason, Professor at St Peter’s College and the Mathematical Institute and myself (Jean-Philippe Nicolas), Professor at the LMBA. The purpose of the thesis will be to extend conformal scattering theories in the two directions mentioned in the previous paragraph. It is an ambitious project but some steps can be completed without major difficulties; they will enable the student to obtain rather quickly some new results while learning the bases of the theory. We shall start by studying the scattering for the semi-linear, defocusing, conformally invariant wave equation
on the Schwarzschild metric. The decay results obtained by Dafermos and Rodnianski [2] will be sufficient to obtain a complete scattering. We shall then extend the results to semi-linear wave equations with sub-critical nonlinearities that are stronger than the conformally invariant one. Finally, the question of the charges for the Maxwell-wave system on flat spacetime will be tackled.

The thesis will take place for the most part in Brest but we are planning at least two visits, of three months minimum each, of the student to Oxford. I will be present in Oxford for one of those visits.

Références


